28. M. W. Zemansky, Heat and Thermodynamics, McGraw Hill, New York (1968).
29. C. A. Angell, J. Shuppert, and J. C. Tucker, "Anomalous properties of supercooled water. Heat capacity, expansivity and proton magnetic resonance chemical shift from 0° to −38°," J. Phys. Chem., 77, No. 26 (1973).
30. L. D. Landau and E. M. Lifshits, Hydrodynamics [in Russian], Nauka, Moscow (1986).
31. J. N. Johnson, D. B. Hayes, and J. R. Asay, "Equations of state and shock-induced transformations in solid I—solid II—liquid bismuth," J. Phys. Chem. Solids, 35, No. 3 (1974).
32. I. P. Bazarov, Thermodynamics [in Russian], Vysshaya Shkola, Moscow (1976).
33. M. Mellor, "Engineering properties of snow," J. Glaciology, 19, No. 81 (1977).

# SPECTRAL MODELS FOR CASCADE PROCESSES IN HOMOGENEOUS TURBULENCE

A. G. Bershadskii

UDC 532.517.4

There is a long history [1, 2] of simulating energy transport in homogeneous isotropic turbulence by means of spectral-density equations. Each model to some extent reflects the energy transport over the spectrum, but those spectral models usually do not reflect the cascade transport, i.e., sequential transport via nearest neighbors in the spectrum. That specific feature of turbulent transport is closely reflected by a reduction model due to Obukhov, Desnyanskii and Novikov, Gledzer, and so on [3-5]. It is of interest to demonstrate cascade transport directly on a spectral model. Here we propose such a model for homogeneous isotropic turbulence.

The stationary Kolmogorov—Obukhov state (−5/3 law) is obtained in a scale-invariant range, together with the nonstationary state having spectral density $E(k, t) \sim t^{-2}k^{-3}$. In the latter, there is energy transfer from the small-scale pulsations to the large-scale ones, which is usually ascribed to two-dimensional turbulence [6]. That state is observed also in lattice turbulence, which was used to simulate two-dimensional in the [7] experiments. In the dissipative range (in the short-wave limit), the model leads to a spectrum $E(k) \sim \exp - ak$, which with logarithmic accuracy coincides with the Kreichnan—Kuz'min—Patashinskii asymptote [8].

I have calculated the damping for the total pulsation energy and the increase in the integral scale for the initial conditions $E_0(k) \sim k^m \exp -(k/k_0)^2$. The result is $\overline{u}^2 \sim t^{-n}$, $L \sim t^p$, in which $n = 2(1 + m)/(3 + m)$; $p = 2/(3 + m)$; and L is the integral turbulence scale. For m = 1-4 corespondingly, those formulas give n = 1, p = 1/2; n = 1.2, p = 0.4; n = 4/3, p = 1/3; n = 10/7, p = 2/7, i.e., values familiar from experiments and various theories [1, 9-14].

Hypotheses on the vortex interaction in turbulent flows are frequently formulated as spectral transport functions T(E; k, t) in the equation for such transport [1]

$$\partial E(k,\ t)/\partial t = -2\nu k^2 E(k,\ t) + T(E;\ k,\ t) \tag{1}$$

in which T(E; k, t) is a function of k and t and a functional of E(k, t). This incorporates inertial energy transport. The explicit form of that function-functional is unknown. We expand T(E; k, t) as a functional series in powers of E(k, t), and as the inertial effects are nonlinear, the series will be analogous not to a Taylor series but instead to an expansion near a branch point:

$$T(E;\ k, t) = \sum_{n=0}^{\infty} \int_0^{\infty} dk_1 \ldots dk_n G(k;\ k_1, \ldots, k_n)\, E^{1/m}(k_1, t) \ldots E^{1/m}(k_n, t). \tag{2}$$

Here m is a positive number (the algebraic order of the branch point), while $G(k; k_1, \ldots, k_n)$ describes the inertial effects from vortices having scales $k_1^{-1}, \ldots, k_n^{-1}$ on vortices having scales $k^{-1}$.

Donetsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 49-54, May-June, 1990. Original article submitted December 19, 1988.

0021-8944/90/3103-0385$12.50 

If one specifies the order of the reduction or increase in the vortex scale in the cascade process $\alpha$, which is the one most likely to occur, then the maximally coarse cascade type of interaction can be represented by the approximation

$$G(k;\ k_1, \ldots, k_n) = \sum g_{n,p_1,\ldots,p_n}(k)\, \delta\left(\alpha^{p_1}k - k_1\right) \ldots \delta\left(\alpha^{p_n}k - k_n\right), \quad (3)$$

in which $\delta(x)$ is a Dirac delta function and $p_1, \ldots, p_n$ take the values 0, 1, or −1, while $g_{n,p_1,\ldots,p_n}(k, t)$ are certain functions. In each elementary energy-redistribution act, there is self-action, so in each of the terms in Eq. (3), at least one of $p_1, \ldots, p_n$ should be zero.

If E is small enough, one can retain only the leading power in Eq. (2), which is defined by comparison with the Euler model, namely on the basis that the equations of motion for the liquid are quadratically nonlinear. In the terms used in the present study, this represents homogeneity in the functional T(E) of order 3/2. We use that specification and substitute Eq. (3) into Eq. (2) to get in the leading approximation that

$$T = \sum g_{p_1p_2p_3}(k)\, E^{1/2}\left(\alpha^{p_1}k\right) E^{1/2}\left(\alpha^{p_2}k\right) E^{1/2}\left(\alpha^{p_3}k\right). \quad (4)$$

Here $p_1$, $p_2$, and $p_3$ take the values 0, 1, or −1, and in each term in the sum, at least one of $p_1$, $p_2$, $p_3$ should be zero. If we now impose the requirement of scale invariance on the coefficient functions $g_{p_1p_2p_3}(k)$, then this with considerations of dimensions gives

$$T = k^{3/2} \sum a_{p_1p_2p_3} E^{1/2}\left(\alpha^{p_1}k\right) E^{1/2}\left(\alpha^{p_2}k\right) E^{1/2}\left(\alpha^{p_3}k\right) \quad (5)$$

in which $a_{p_1p_2p_3}$ are dimensionless constants. We substitute Eq. (5) into Eq. (1) to get

$$\partial E(k, t)/\partial t = k^{3/2} \sum a_{p_1p_2p_3} E^{1/2}\left(\alpha^{p_1}k\right) E^{1/2}\left(\alpha^{p_2}k\right) E^{1/2}\left(\alpha^{p_3}k\right) - 2\nu k^2 E(k).$$

As energy should be conserved in the absence of viscosity, i.e.,

$$\int_0^\infty k^{3/2} \sum a_{p_1p_2p_3} E^{1/2}\left(\alpha^{p_1}k\right) E^{1/2}\left(\alpha^{p_2}k\right) E^{1/2}\left(\alpha^{p_3}k\right) = 0,$$

we get

$$\partial E(k, t)/\partial t = k^3 a_1 [E^{1/2}(k)E(\alpha k) - \alpha^{-5/2}E(k)E^{1/2}(k/\alpha)] + k^{3/2}a_2[E(k)E^{1/2}(\alpha k) - \alpha^{-5/2}E^{1/2}(k)E(k/\alpha)] - 2\nu k^2 E(k) \quad (6)$$

in which $a_1$ and $a_2$ are dimensionless constants.

It is useful to recall the general principles from which the reduced Obukhov–Desnyanskii–Novikov equations have been derived [4]: 1) quadratic nonlinearity with respect to the velocity pattern; 2) scale invariance for the dimensionless coefficients; 3) interaction occurring directly only by nearest neighbors in the spectrum; and 4) a quadratic integral in the nonviscous case.

We have in fact used those principles in deriving Eq. (6). Of course, the spectral model is cruder than a reduction one, but it is convenient for operating directly with the spectral energy density.

Firstly, as with the reduction model, we derive the stationary solution to Eq. (6) in the nonviscous approximation, i.e., the solution to

$$k^{3/2}a_1[E^{1/2}(k)E(\alpha k) - \alpha^{-5/2}E(k)E^{1/2}(k/\alpha)] + k^{3/2}a_2[E(k)E^{1/2}(\alpha k) - \alpha^{-5/2}E^{1/2}(k)E(k/\alpha)] = 0.$$

In addition to the trivial solution E(k) = 0, there is a solution

$$E(k) \sim k^{-5/3}, \quad (7)$$

i.e., the spectral energy density as a Kolmogorov–Obukhov law for the inertial interval. If we incorporate the viscosity in the stationary situation, i.e., use the equation

$$k^{3/2}a_1[E^{1/2}(k)E(\alpha k) - \alpha^{-5/2}E(k)E^{1/2}(k/\alpha)] + k^{3/2}a_2[E(k)E^{1/2}(\alpha k) - \alpha^{-5/2}E^{1/2}(k)E(k/\alpha)], \quad (8)$$

one can determine the short-wave asymptote E(k). For two-fold reduction (union) with $\alpha = 1/2$ [4], that asymptote to Eq. (8) is

$$E(k) \sim k \exp(-ak) \quad (9)$$

in which $a$ is a constant. That asymptote for $k \to \infty$ coincides with logarithmic accuracy with the standard Kreichnan–Kuz'min–Patashinskii [8] short-wave asymptote. To describe E(k) throughout the viscoinertial interval, including the inertial approximation (7) and the (9) short-wave asymptote, it is necessary to solve Eq. (8) without discarding the viscous term and without using the asymptotic approximation $k \to \infty$.

Interest attaches to the reversibility in the absence of viscosity, namely the reversibility of Eq. (6) when the viscous term is absent. When t is replaced by −t (time reversal), the vortex break-up must be replaced by union. In the reversibility analysis, the replacement of t by −t must be accompanied by the replacement of α by $\alpha^{-1}$. Then Eq. (6) without the viscous term is seen to be invariant under those transformations if

$$a_1 = a_2 = a_0\alpha^{5/2}. \quad (10)$$

Then this spectral model is reversible in the nonviscous case if Eq. (10) is obeyed.

In the scale-invariant range, where E is a power function of k, Eq. (6) in the nonviscous approximation has the nonstationary solution

$$E(k,\ t) = At^{-2}k, \quad (11)$$

if $a_0 < 0$ for $\alpha < 1$ or $a_0 > 0$ for $\alpha > 1$. Then

$$A = 4\alpha/[a_0^2(1 + \alpha^{3/2} - \alpha^2 - \alpha^{7/2})^2].$$

A solution of Eq. (11) type is stable under small perturbations of $\delta(t)k^{-3+\varepsilon}$ type, in which $\delta(0)$ and ε are sufficiently small numbers.

In a reduction model, the $E \sim k^{-3}$ spectrum is related to a nonviscous entrophy integral [3] (when the nonviscous energy invariant is violated). The −3 spectrum is possible also in a model with energy conservation but is substantially nonstationary: $E \sim t^{-2}$. An Eq. (11) spectrum has been discussed in relation to two-dimensional turbulence (see a special collection on this [6]), and has also been observed as $E \sim t^{-2}$ and $E \sim k^{-3}$ [7]. In the [7] experiments, the lattice flow of a highly conducting liquid (mercury) was used in a transverse magnetic field (B = 0.68 T), which simulates two-dimensional turbulence in a plane normal to the field induction vector [15].

The direction of the energy redistribution over the spectrum is of interest [3, 16, 17]. We integrate both parts of Eq. (6) without the viscosity with respect to k' from 0 to k:

$$\partial \int_0^k E(k')\,dk'/\partial t = a_0\left\{\int_0^k dk'k'^{3/2}\left[\alpha^{5/2}E^{1/2}(k')\,E(\alpha k') - E(k')\,E^{1/2}(k')\right] + \right. \\ \left. + \int_0^k dk'k'^{3/2}\left[\alpha^{5/2}E(k')\,E^{1/2}(k') - E^{1/2}(k')\,E(k'/\alpha)\right]\right\}. \quad (12)$$

From Eq. (12) with $\alpha < 1$ we get

$$\partial \int_0^k E(k')\,dk'/\partial t = -a_0 \int_{\alpha k}^k dk'k'^{3/2}\left[E^{1/2}(k'/\alpha)\,E(k') + E^{1/2}(k')\,E(k'/\alpha)\right]. \quad (13)$$

As $E \geq 0$, Eq. (13) gives us that for $a_0 > 0$, $\int_0^k E(k')dk'$ decreases monotonically over time in the nonviscous approximations for $a_0 < 0$, while it increases monotonically with t for $a_0 < 0$. For $\alpha > 1$

$$\partial \int_0^k E(k')\,dk'/\partial t = a_0 \int_k^{\alpha k} dk'k'^{3/2}\left[E^{1/2}(k'/\alpha)\,E(k') + E^{1/2}(k')\,E(k'/\alpha)\right]$$

and the conclusions are the opposite of those for the previous situation ($\alpha < 1$).

As the total energy $\int_0^k E(k)\,dk$ is conserved in the nonviscous approximation, with $\int_0^\infty E \times (k')dk'$ increasing monotonically in time, there is energy transfer from the small-scale pulsations to the large-scale ones, and vice versa, when $\int_0^k E(k)dk$ decreases monotonically.

In the situation where Eq. (11) applies, one should have the relation $\alpha < 1 \Rightarrow a_0 < 0$, $\alpha > 1 \Rightarrow a_0 > 0$, so in any case (for $\alpha < 1$ or for $\alpha > 1$) the energy is transferred from the small-scale pulsations to the large-scale ones (compare the analogous effect in the theory of two-dimensional turbulence [16, 17]).

If we abandon the requirement that Eq. (4) provides energy conservation in the nonviscous case, one can use it to describe the damping for the total pulsation energy, i.e., Eq. (4) is considered as applicable only to the energy-bearing range and is of pure sink type. Then in that range

$$\partial E(k,t)/\partial t = k^{3/2}\sum a_{p_1p_2p_3}E^{1/2}\left(\alpha^{p_1}k\right)E^{1/2}\left(\alpha^{p_2}k\right)E^{1/2}\left(\alpha^{p_3}k\right).$$

We consider an initial condition for E(k, t):

$$E_0(k) = E(k,\,0) = k^m \exp - (k/k_0)^2, \tag{14}$$

as usually employed in numerical models for homogeneous turbulence. We take $k_0$ as such that the exponential factor in Eq. (14) is appreciable only outside the energy-bearing wavenumber range, which means that the initial condition in that range is approximately $E_0(k) \simeq k_m$. We make the substitution $E(k, t) = E'(k, t)k^m$, and then the initial condition for $E' \times (k, t)$ in the energy-bearing range will be $E_0'(k) = 1$, and the equation is

$$\partial E(k,t)/\partial t = k^{(3+2m)/2}\sum a_{p_1p_2p_3}E'^{1/2}\left(\alpha^{p_1}k\right)E'^{1/2}\left(\alpha^{p_2}k\right)E'^{1/2}\left(\alpha^{p_3}k\right).$$

As $E_0'(k) = 1$ (is independent of k in the energy-bearing interval), we introduce the self-similar variable

$$\tau = tk^{(3+2m)/2}. \tag{15}$$

It is clear that $E'(k, t) = E'(\tau)$ in the energy-bearing interval, and the total pulsation energy is $\bar{u}^2 = (3/2)\int_0^\infty E(k)dk$. As that interval contains virtually all the energy, we have approximately that

$$\bar{u}^2 = \frac{3}{2}\int_0^\infty k^m E'(\tau)\,dk. \tag{16}$$

We replace k by $\tau$ in the integral to get from Eq. (16) that

$$\bar{u^2} \sim t^{-n},\; n = 2(1+m)/(3+m). \tag{17}$$

We calculate also the integral turbulence scale (longitudinal [1]):

$$\mathrm{L} = \frac{3\pi}{4}\int_0^\infty E(k)\,k^{-1}\,dk \Big/ \int_0^\infty E(k)\,dk.$$

We convert to the Eq. (15) $\tau$ to get

$$L \sim t^p,\; p = 2/(3+m). \tag{18}$$

We get n = 1 and p = 1/2 from Eqs. (17) and (18) for m = 1. Those values of n and p are familiar from the theory and from experiments on hydrodynamic arrays [1].

Also, n = 1.2 and p = 0.4 for m = 2, which was first observed in Uberoi's lattice experiment [1], while we may note [12] amongst recent ones. In [14], simulation of homogeneous turbulence with Eq. (14) as initial condition and m = 2 gave n = 1.2. Amongst the theoretical or semiempirical studies in which n = 1.2 and p = 0.4, we may note [10].

Also, n = 4/3 and p = 1/3 for m = 3. I have not found any theoretical derivation of those values in the literature. Of the experimental papers, we may note [11].

Finally, n = 10/7 and p = 2/7 for m = 4, which were first derived by Kolomogorov from theory with the use of a Loitsyanskii invariant [1, 18] (this is also related to the value m = 4 [9]), which was observed by experiment in [19].

I am indebted to A. S. Ginevskii and Yu. P. Sosedko for a discussion of the experiment.

## LITERATURE CITED

1. A. S. Monin and A. M. Yaglom, Statistical Hydromechanics [in Russian], Vol. 2, Nauka, Moscow (1967).
2. R. Deissler, "Turbulent processes and simple closure schemes," in: Turbulence Principles and Applications [Russian translation], Mir, Moscow (1980).
3. E. B. Gledzer, F. V. Dolzhanskii, and A. M. Obukhov, Hydrodynamic-Type Systems and Their Applications [in Russian], Nauka, Moscow (1981).
4. V. N. Desnyanskii and E. A. Novikov, "Simulating cascade processes in turbulent flows," Prikl. Mat. Mekh., 38, No. 3 (1974).
5. E. B. Gledzer, "The 2/3 law in turbulence theory and estimation of the constant appearing there by hydrodynamic-equation reduction," Zh. Éksp. Teor. Fiz., 91, No. 3 (1986).
6. Turbulence Bidimensionnelle, J. de Méc., Num. Spécial (1983).
7. J. Sommeria, "Two-dimensional behavior of MHD fully developed turbulence," J. de Méc., No. 2 (1983).
8. M. M. Dubovnikov and V. I. Tatarskii, "Calculating the asymptote for the locally isotropic turbulence spectrum in the viscous range," Zh. Éksp. Teor. Fiz., 93, No. 6 (1987).
9. G. Comte-Bellott and S. Corrsin, "The use of a contraction to improve isotropy of grid-generated turbulence," J. Fluid Mech., 25, No. 4 (1966).
10. P. G. Saffman, "A phenomenological theory for calculating turbulent shear flows," in: Strange Attractors [Russian translation], Mir, Moscow (1981).
11. A. S. Ginevskii, A. V. Kolesnikov, and L. N. Ukhanova, "Degeneration in flow turbulence behind a two-row grid composed of cylinders with opposite motions for the rows," Izv. AN SSSR, MZhG, No. 3 (1979).
12. K. R. Sreenivasan, S. Tavoularis, et al., "Temperature fluctuations and scales in grid-generated turbulence," J. Fluid Mech., 100, No. 3 (1980).
13. W. Schumann, G. Gretzbach, and L. Kleiser, "Direct methods in the numerical simulation of turbulent flows," in: Turbulent-Flow Computation Methods [Russian translation], Mir, Moscow (1984).
14. T. Tatsumi and S. Kida, "The modified cumulant expansion for isotropic turbulence at large Reynolds number," J. Phys. Soc. Jpn., 49, No. 5 (1980).
15. J. Sommeria and R. Moreau, "Why, how and when MHD turbulence becomes two-dimensional," J. Fluid Mech., 118, 507 (1982).
16. A. P. Mirabel' and A. S. Monin, "Two-dimensional turbulence," in: Advances in Mechanics, Vol. 2, No. 3, Poland (1979).
17. D. Montgomery, "Cascades and inverse cascades in fluids and magnetofluids," Res. Rept., Inst. Plasma Phys., Nagoya Univ., No. 670 (1984).
18. A. N. Kolmogorov, "Isotropic-turbulence degeneration in an incompressible liquid," Dokl. Akad. Nauk SSSR, 31, No. 6 (1941).
19. G. I. Derbunovich, A. S. Zemskaya, et al., "Use of grids for controlling the structure of turbulent flows in wind tunnels," Uchen. Zap. TsAGI, 13, No. 1 (1982).